C3 JAN 11

1. (a) Express $7\cos x - 24\sin x$ in the form $R\cos(x + \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.
Give the value of α to 3 decimal places. (3)

(b) Hence write down the minimum value of $7\cos x - 24\sin x$. (1)

(c) Solve, for $0 \leqslant x < 2\pi$, the equation

$$7\cos x - 24\sin x = 10$$

giving your answers to 2 decimal places. (5)

a) $R\cos(x+\alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$

$R\cos(x+\alpha) = 7\cos x - 24\sin x$

$\frac{R\sin\alpha = 24}{R\cos\alpha = 7} \Rightarrow \tan\alpha = \frac{24}{7} \Rightarrow \alpha = 1.287^c$

b) $R^2 = 24^2 + 7^2 \Rightarrow R = 25$

$25\cos(x + 1.287)$ $\quad$ $\min = -25$

c) $25\cos(x + 1.287) = 10$

$x + 1.287 = \cos^{-1}\left(\frac{10}{25}\right) = 1.159, 5.1239, 7.443$

$-1.287 \Rightarrow x = 3.84^c, 6.16^c$

2. (a) Express

$$\frac{4x-1}{2(x-1)} - \frac{3}{2(x-1)(2x-1)}$$

as a single fraction in its simplest form. (4)

Given that

$$f(x) = \frac{4x-1}{2(x-1)} - \frac{3}{2(x-1)(2x-1)} - 2, \quad x > 1,$$

(b) show that

$$f(x) = \frac{3}{2x-1}$$

(2)

(c) Hence differentiate $f(x)$ and find $f'(2)$. (3)

a) $\frac{(4x-1)(2x-1)-3}{2(x-1)(2x-1)} = \frac{8x^2-6x+1-3}{2(x-1)(2x-1)}$

$= \frac{\cancel{2}(4x+1)\cancel{(x-1)}}{\cancel{2}\cancel{(x-1)}(2x-1)} = \frac{4x+1}{2x-1}$

b) $f(x) = \frac{4x+1}{2x-1} - \frac{2}{1}\frac{\times(2x-1)}{\times(2x-1)} = \frac{4x+1-4x+2}{2x-1}$

$f(x) = \frac{3}{2x-1}$ #

c) $f(x) = 3(2x-1)^{-1} \Rightarrow f'(x) = -3\times 2(2x-1)^{-2}$

$f'(x) = \frac{-6}{(2x-1)^2} \Rightarrow f'(2) = \frac{-6}{3^2} = -\frac{2}{3}$

3. Find all the solutions of

$$2\cos 2\theta = 1 - 2\sin\theta$$

in the interval $0 \leq \theta < 360°$.

(6)

$2 - 4\sin^2\theta = 1 - 2\sin\theta$

$\Rightarrow \; 4\sin^2\theta - 2\sin\theta - 1 = 0$

$\Rightarrow \; \sin^2\theta - \frac{1}{2}\sin\theta = \frac{1}{4}$

$\Rightarrow \; \left(\sin\theta - \frac{1}{4}\right)^2 = \frac{1}{4} + \frac{1}{16} = \frac{5}{16}$

$\Rightarrow \; \sin\theta = \frac{1}{4} \pm \frac{\sqrt{5}}{4} = \frac{1 \pm \sqrt{5}}{4}$

$\theta = \sin^{-1}\left(\frac{1+\sqrt{5}}{4}\right) = \underline{54°}, \underline{126°}$

$\theta = \sin^{-1}\left(\frac{1-\sqrt{5}}{4}\right) = \cancel{-18°}, \underline{198°}, \underline{342°}$

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4. Joan brings a cup of hot tea into a room and places the cup on a table. At time t m after Joan places the cup on the table, the temperature, θ°C, of the tea is modelled by equation

$$\theta = 20 + Ae^{-kt},$$

where A and k are positive constants.

Given that the initial temperature of the tea was 90°C,

(a) find the value of A.

(2)

The tea takes 5 minutes to decrease in temperature from 90°C to 55°C.

(b) Show that $k = \frac{1}{5}\ln 2$.

(3)

(c) Find the rate at which the temperature of the tea is decreasing at the instant when $t = 10$. Give your answer, in °C per minute, to 3 decimal places.

(3)

a) $90 = 20 + A \Rightarrow \underline{A = 70}$

b) $55 = 20 + 70e^{-5k} \Rightarrow 35 = 70e^{-5k}$

$\Rightarrow e^{-5k} = \frac{1}{2} \Rightarrow -5k = \ln\left(\frac{1}{2}\right) = \ln(2^{-1}) = -\ln 2$

$\Rightarrow k = \frac{1}{5}\ln 2$ #

c) $\frac{d\theta}{dt} = \left(70 \times -\frac{1}{5}\ln 2\right)e^{-\frac{1}{5}\ln 2\, t} = -14\ln 2\, e^{-\frac{1}{5}\ln 2\, t}$

$t = 10 \quad \frac{d\theta}{dt} = (-14\ln 2)e^{-2\ln 2} = -2.426$

$\Rightarrow$ decreasing at a rate of $\underline{2.426°C/\text{min}}$

5

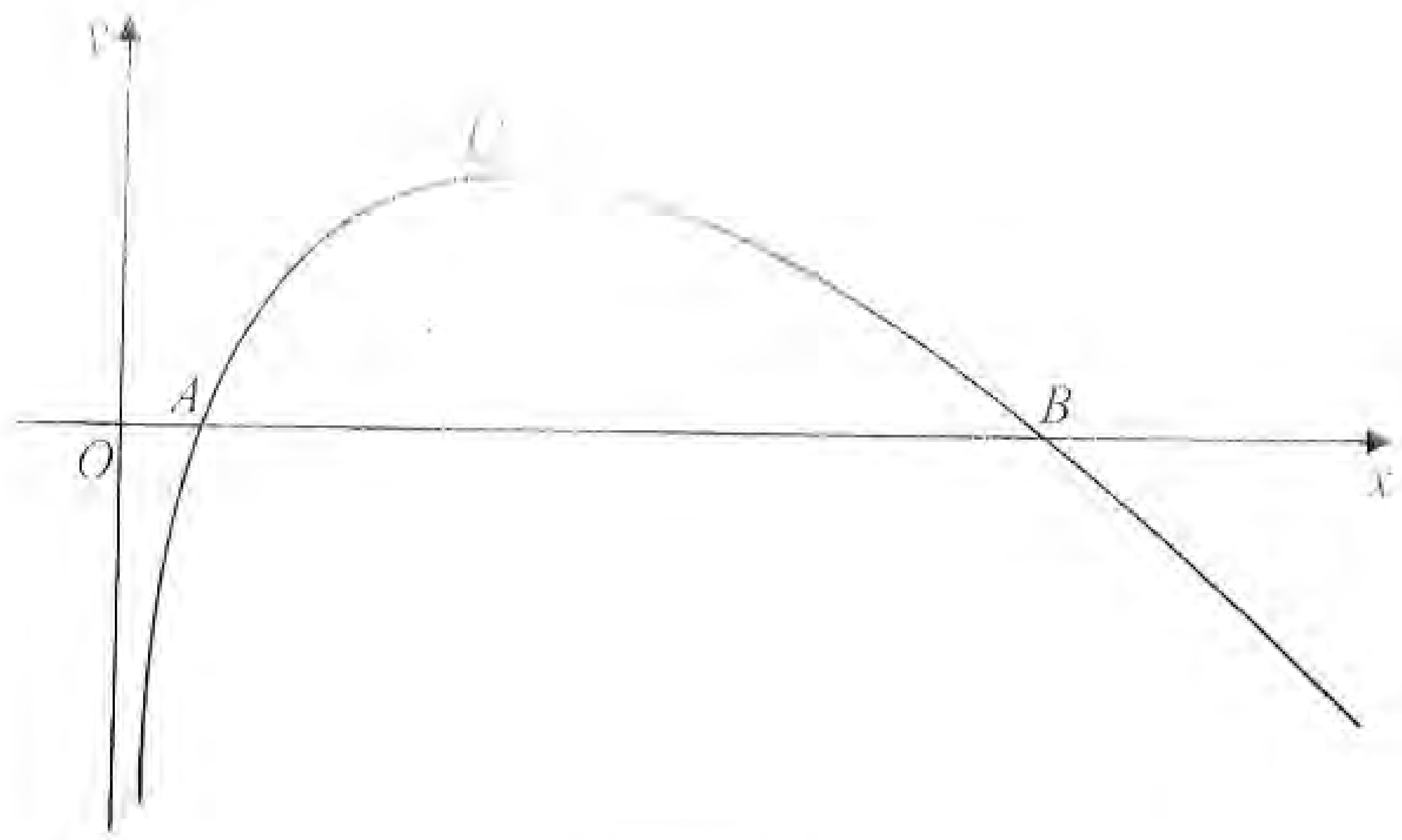


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = f(x)$, where

$$f(x) = (8 - x)\ln x, \quad x > 0$$

The curve cuts the x-axis at the points A and B and has a maximum turning point at Q, as shown in Figure 1.

(a) Write down the coordinates of A and the coordinates of B.

(2)

(b) Find $f'(x)$.

(3)

(c) Show that the x-coordinate of Q lies between 3.5 and 3.6

(2)

(d) Show that the x-coordinate of Q is the solution of

$$x = \frac{8}{1 + \ln x}$$

(3)

To find an approximation for the x-coordinate of Q, the iteration formula

$$x_{n+1} = \frac{8}{1 + \ln x_n}$$

is used.

(e) Taking $x_0 = 3.55$, find the values of x_1, x_2 and x_3. Give your answers to 3 decimal places.

(3)

5a) $(8-x)\ln x = 0$ when $x = 8, x = 1$ $\underline{A(1,0)}$ $\underline{B(8,}$

b) $f'(x) = (8-x)'\ln x + (8-x)(\ln x)'$

$f'(x) = -\ln x + \frac{8-x}{x}$

c) $f'(3.5) = 0.033 \Rightarrow$ Increasing function at $x = 3.5$

$f'(3.6) = -0.059 \Rightarrow$ decreasing function at $x = 3.6$

$\Rightarrow$ turning point between 3.5 and 3.6.

d) $f'(x) = 0 \Rightarrow \frac{8-x}{x} = \ln x \Rightarrow 8 - x = x\ln x$

$\Rightarrow 8 = x + x\ln x \Rightarrow 8 = x(1 + \ln x) \Rightarrow x = \frac{8}{1+\ln x}$

e) $x_0 = \underline{3.55}$; $x_1 = \underline{3.529}$; $x_2 = \underline{3.538}$; $x_3 = \underline{3.53}$

6. The function f is defined by

$$f: x \mapsto \frac{3-2x}{x-5}, \quad x \in \mathbb{R}, \; x \neq 5$$

(a) Find $f^{-1}(x)$.

(3)

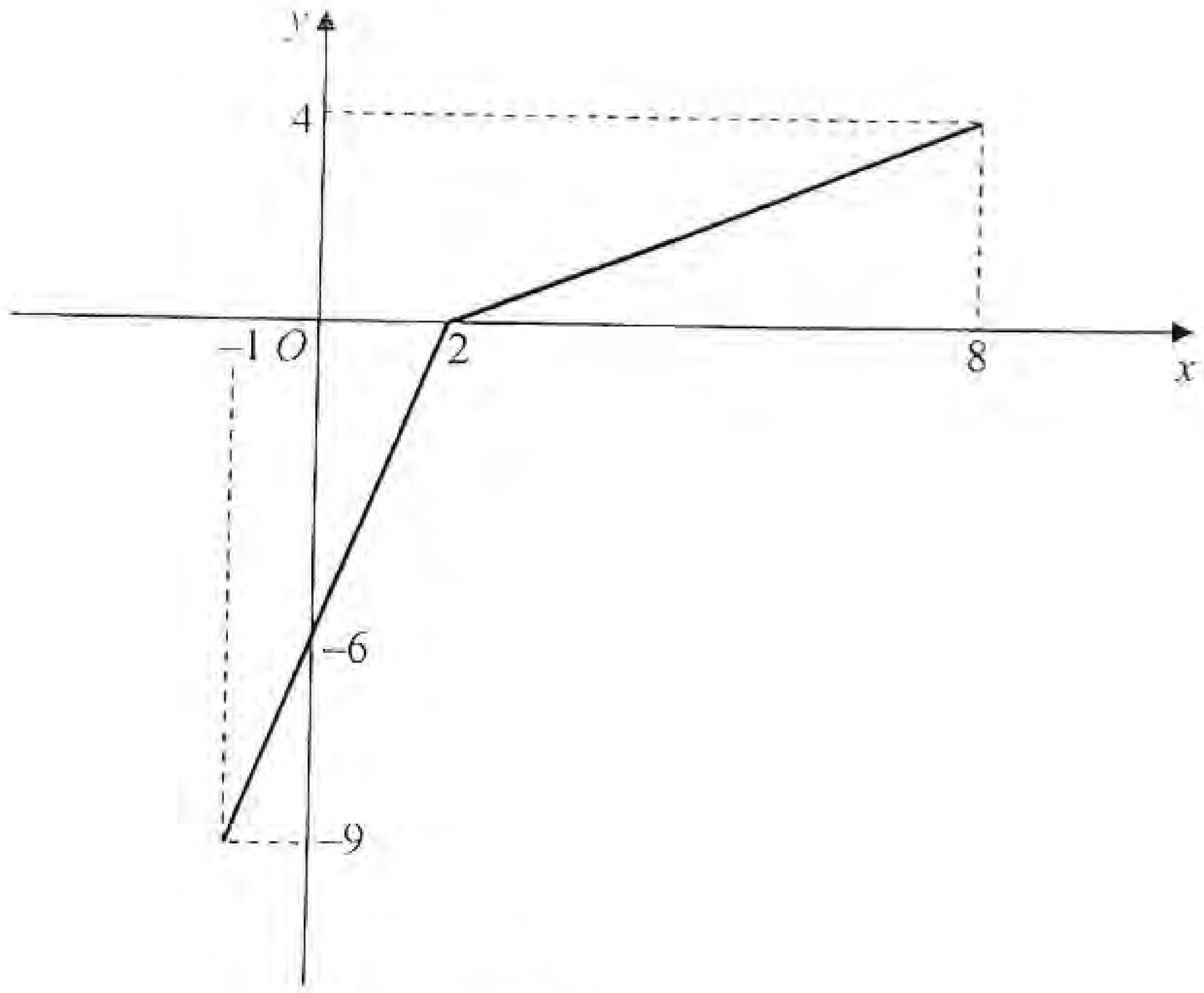


Figure 2

The function g has domain $-1 \leqslant x \leqslant 8$, and is linear from $(-1, -9)$ to $(2, 0)$ and from $(2, 0)$ to $(8, 4)$. Figure 2 shows a sketch of the graph of $y = g(x)$.

(b) Write down the range of g.

(1)

(c) Find gg(2).

(2)

(d) Find fg(8).

(2)

(e) On separate diagrams, sketch the graph with equation

(i) $y = |g(x)|$,

(ii) $y = g^{-1}(x)$.

Show on each sketch the coordinates of each point at which the graph meets or cuts the axes.

(4)

(f) State the domain of the inverse function g^{-1}.

(1)

6) $x = \frac{3-2y}{y-5} \Rightarrow xy - 5x = 3 - 2y \Rightarrow xy + 2y = 5x + 3$

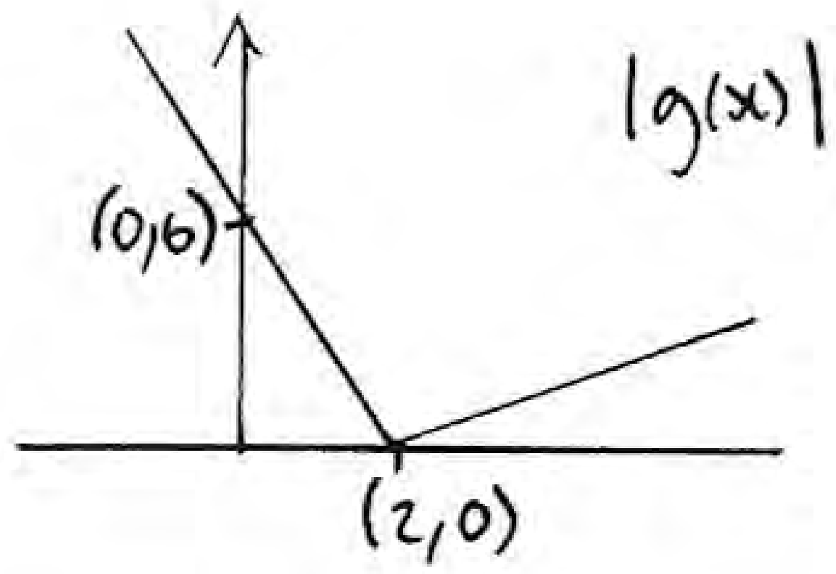
$\Rightarrow (x+2)y = 5x+3 \Rightarrow y = f^{-1}(x) = \frac{5x+3}{x+2}$

b) $-9 \leq g(x) \leq 4 \quad g(x) \in \mathbb{R}$.

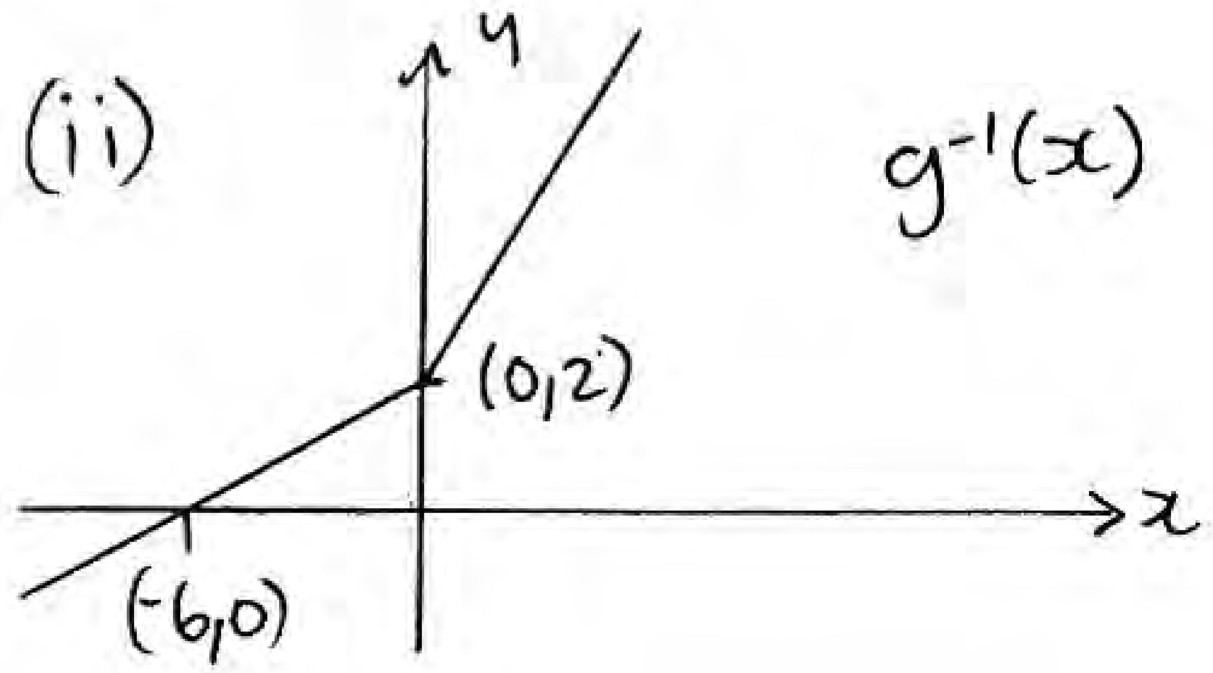
c) $gg(2) = g(0) = \underline{-6}$

d) $fg(8) = f(4) = \frac{3-8}{4-5} = \frac{-5}{-1} = \underline{5}$

e)



(ii)



f) domain g^{-1} = range $g \Rightarrow \underline{-9 \leq x \leq 4} \quad x \in \mathbb{R}$

7. The curve C has equation

$$y = \frac{3+\sin 2x}{2+\cos 2x}$$

(a) Show that

$$\frac{dy}{dx} = \frac{6\sin 2x + 4\cos 2x + 2}{(2+\cos 2x)^2}$$

(4)

(b) Find an equation of the tangent to C at the point on C where $x = \frac{\pi}{2}$.

Write your answer in the form $y = ax + b$, where a and b are exact constants.

(4)

a) $\frac{dy}{dx} = \frac{(3+\sin 2x)'(2+\cos 2x) - (3+\sin 2x)(2+\cos 2x)'}{(2+\cos 2x)^2}$

$= \frac{2\cos 2x(2+\cos 2x) - (3+\sin 2x)(-2\sin 2x)}{(2+\cos 2x)^2}$

$= \frac{4\cos 2x + 2\cos^2 2x + 6\sin 2x + 2\sin^2 2x}{(2+\cos 2x)^2}$

$= \frac{6\sin 2x + 4\cos 2x + 2\underbrace{(\sin^2 2x + \cos^2 2x)}_{=1}}{(2+\cos 2x)^2}$

$= \frac{6\sin 2x + 4\cos 2x + 2}{(2+\cos 2x)^2}$ #

b) $x = \frac{\pi}{2}$; $y = 3$; $m_t = -2$

$y - 3 = -2(x - \frac{\pi}{2}) \Rightarrow y = -2x + \pi + 3$

8. (a) Given that

$$\frac{d}{dx}(\cos x) = -\sin x$$

show that $\frac{d}{dx}(\sec x) = \sec x \tan x$.

(3)

Given that

$$x = \sec 2y$$

(b) find $\frac{dx}{dy}$ in terms of y.

(2)

(c) Hence find $\frac{dy}{dx}$ in terms of x.

(4)

a) $\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{d}{dx}(\cos x)^{-1}$

$= -(\cos x)^{-2} \times -\sin x = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \times \frac{\sin x}{\cos x}$

$= \sec x \tan x$ #

b) $x = \sec 2y$

$\frac{dx}{dy} = 2\sec 2y \tan 2y$

c) $\frac{dy}{dx} = \frac{1}{2\sec 2y \tan 2y}$

$\sec 2y = x$

$\tan^2 2y + 1 = \sec^2 2y$

$\tan^2 2y = x^2 - 1$

$\tan 2y = \sqrt{x^2-1}$

$\frac{dy}{dx} = \frac{1}{2x\sqrt{x^2-1}}$